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NON-LINEAR ANALYSIS OF PRESTRETCHED CIRCULAR MEMBRANE AND A MODIFIED ITERATION TECHNIQUE

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Abstract—Using a non-linear theory, the deflection of a prestretched circular membrane subjected to ponding pressure is analyzed. The analysis may serve as a typical example of those studies where linear theories are not admissible for obtaining proper responses by an elastic system. A modified iteration technique which yields fast convergence is presented to simplify the solution effort. Four cases which are linked to the four different levels of problem-solving are examined and compared. A better understanding of the non-linear behavior of a prestretched circular membrane is reached.

INTRODUCTION

In a recent paper, Kerr and Coffin (1990) studied the ponding deflection of a membrane strip. They found that the non-linear homogeneous governing equation for the membrane strip has a bifurcated solution when the load parameter is relatively large, and that the first eigenvalue of the linear analysis determines merely the bifurcation point. Based on this knowledge, it is seen that deflection of the system must not only *exist* beyond the bifurcation point, but it also grows with an increase of the load parameter. Thus, the analysis shed light on the matter, even though the ponding system discussed is of the simplest form (membrane "strip").

The current paper presents a non-linear analysis of the deflection of a prestretched circular membrane subjected to the weight of a liquid filling the space created by the deflection of the membrane (Fig. 1). Hence, the pressure applied on the prestretched circular membrane depends on the deflection of the membrane. Prestretched circular membranes have many practical applications: thin films such as silicon nitride membranes are used in the electronics industry as well as other industries, but the mechanical properties of such membranes have not been studied in detail.

Also presented herein is a modified iteration technique which can yield approximate but accurate solutions for the two coupled non-linear differential equations for prestretched membranes. The modified iteration technique has the advantages of simplicity and effectiveness, making it well suited for practical usage (Liu, 1984). As a more general analysis, the weight of the membrane itself, which must have some influence on the examined ponding deflection, is incorporated into the treatment, and the modification introduced by the membrane weight is also explored, giving a more realistic picture of the ponding system.

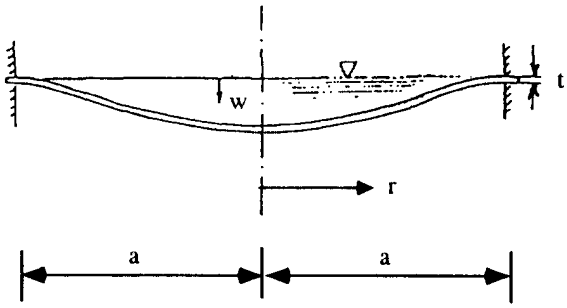

Fig. 1. A prestretched circular membrane subject to liquid load.

MATHEMATICAL FORMULATION OF THE PROBLEM

Figure 1 is a schematic diagram of a circular elastic membrane under a ponding load. It is assumed that the membrane has been prestretched by an initial tension T_0 (force per unit length). Let the radius of the membrane be a, and the thickness, t. We denote Young's modulus and Poisson's ratio for the membrane material as E and ν. To fit the membrane geometry, a cylindrical coordinate system (r, θ, z) is adopted as shown.

Due to the axial symmetry of the membrane deflection, only two components of displacement, i.e. radial displacement $u(r)$ and axial displacement (deflection) $w(r)$, exist and are independent of the circumferential coordinate θ.

The existing membrane strain components (Timoshenko and Woinowsky-Krieger, 1959; Chia, 1980) are:

$$\varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \tag{1a}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{1b}$$

among which ε_r is non-linearly linked to the possibly large deflection w. Using Hooke's law and taking into account the initial tension, the membrane internal forces can be expressed as

$$\begin{aligned} T_r &= T_0 + \tilde{T}_r \\ T_\theta &= T_0 + \tilde{T}_\theta, \end{aligned} \tag{2}$$

where

$$\tilde{T}_r = \frac{Et}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) \tag{3}$$

$$\tilde{T}_\theta = \frac{Et}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r).$$

The vertical load per unit area on the membrane is

$$q = \gamma w(r) + \gamma_0 t \tag{4}$$

in which γ is the specific weight of the liquid and γ_0 the specific weight of the membrane. Referring to the non-linear deflection theory of thin plates (Timoshenko and Woinowsky-Krieger, 1959; Chia, 1980), we may, in a parallel manner, put the differential equation for the membrane equilibrium as

$$\frac{dT_r}{dr} + \frac{1}{r}(T_r - T_\theta) = 0$$

$$T_r \frac{d^2 w}{dr^2} + T_\theta \cdot \frac{1}{r}\frac{dw}{dr} + q(r) = 0$$

or

$$\frac{d\tilde{T}_r}{dr} + \frac{1}{r}(\tilde{T}_r - \tilde{T}_\theta) = 0 \tag{5a}$$

$$T_0\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = -\gamma w - \gamma_0 t - \tilde{T}_r \frac{d^2 w}{dr^2} - \tilde{T}_\theta \cdot \frac{1}{r}\frac{dw}{dr}. \tag{5b}$$

As for the membrane boundary conditions, they are

$$r = a: \quad u = 0, \quad w = 0. \tag{6}$$

Furthermore, all quantities concerned must be finite at $r = 0$.

STRESS FUNCTION AND NON-DIMENSIONALIZATION OF EQUATIONS

As in the linear theory of thin plates, introducing a stress function to simplify the structure of equations is desirable. By expressing internal forces $\tilde{T}_r$, $\tilde{T}_\theta$ in terms of a stress function $\phi(r)$:

$$\tilde{T}_r = \frac{1}{r}\frac{d\phi}{dr}, \quad \tilde{T}_\theta = \frac{d^2\phi}{dr^2} \tag{7}$$

equilibrium eqn (5a) is satisfied identically, and eqn (5b) becomes

$$T_0\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = -\gamma w - \gamma_0 t - \frac{1}{r}\frac{d\phi}{dr}\frac{d^2 w}{dr^2} - \frac{d^2\phi}{dr^2}\cdot\frac{1}{r}\frac{dw}{dr}. \tag{8a}$$

To obtain further equations for the determination of w and ϕ, one must consider the compatibility of internal forces. From strain expressions (1a, b), we obtain the condition of compatibility between ε_r and ε_θ:

$$\varepsilon_r - \frac{d}{dr}(r\varepsilon_\theta) = \frac{1}{2}\left(\frac{dw}{dr}\right)^2. \tag{9}$$

After invoking relation (3) and definition (7), this condition is equivalent to

$$(1+\nu)\left(\frac{1}{r}\frac{d\phi}{dr} - \frac{d^2\phi}{dr^2}\right) - r\frac{d}{dr}\left(\frac{d^2\phi}{dr^2} - \frac{\nu}{r}\frac{d\phi}{dr}\right) = \frac{Et}{2}\left(\frac{dw}{dr}\right)^2$$

or, after differentiation,

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right) = -\frac{Et}{2}\cdot\frac{1}{r}\frac{d}{dr}\left(\frac{dw}{dr}\right)^2. \tag{8b}$$

Equations (8a, b) are the two coupled non-linear differential equations for the two basic field variables, w and ϕ, of the problem.

Correspondingly, by means of eqns (1b), (3) and (7), boundary conditions (6) are rewritten as

$$r = a: \quad w = 0, \quad \frac{d^2\phi}{dr^2} - \frac{\nu}{r}\frac{d\phi}{dr} = 0. \tag{10}$$

In what follows, for conciseness of treatment, non-dimensional variables $\bar{w} = w/a$, $\psi = \phi/(Eta^2)$ as well as the non-dimensional coordinate $\rho = r/a$ are used. With these new variables and coordinates, eqns (8a, b) and boundary conditions (10) become

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right)\left(\frac{d^2\psi}{d\rho^2} + \frac{1}{\rho}\frac{d\psi}{d\rho}\right) = -\frac{1}{2\rho}\frac{d}{d\rho}\left(\frac{d\bar{w}}{d\rho}\right)^2 \tag{11a}$$

$$\frac{d^2\bar{w}}{d\rho^2} + \frac{1}{\rho}\frac{d\bar{w}}{d\rho} = -\beta - \lambda\bar{w} - \varepsilon\left(\frac{1}{\rho}\frac{d\psi}{d\rho}\cdot\frac{d^2\bar{w}}{d\rho^2} + \frac{d^2\psi}{d\rho^2}\cdot\frac{1}{\rho}\frac{d\bar{w}}{d\rho}\right) \tag{11b}$$

$$\rho = 1: \quad \bar{w} = 0 \tag{12a}$$

$$\rho = 1: \quad \frac{d^2\psi}{d\rho^2} - \frac{\nu}{\rho}\frac{d\psi}{d\rho} = 0, \tag{12b}$$

where $\beta = \gamma_0 ta/T_0$ is the parameter for membrane weight, $\lambda = \gamma a^2/T_0$ is the parameter for liquid load and $\varepsilon = Et/T_0$ is the parameter which indicates the degree of non-linearity in eqn (11b).

MODIFIED ITERATION TECHNIQUE

To search for an acceptable approximate solution for the two coupled nonlinear differential eqns (11a, b) by an iteration scheme, a linear counterpart of eqn (11b)

$$\frac{d^2\bar{w}}{d\rho^2} + \frac{1}{\rho}\frac{d\bar{w}}{d\rho} = -\beta \tag{13}$$

is first examined. From eqn (13) and the boundary condition (12a), we take

$$\bar{w} = \bar{w}_0(1-\rho^2) \tag{14}$$

as the first approximate form for the membrane deflection, with $\bar{w}_0$ representing the central deflection. We note that in conducting the subsequent iteration, $\bar{w}_0$ is seen as an undetermined parameter, to which a proper value will be assigned later by consideration of convergence. It is a modified and more effective iteration strategy.

Substituting eqn (14) into the right-hand side of eqn (11a) and integrating the equation, in conjunction with boundary condition (12b) for ψ, yields

$$\psi = \bar{w}_0^2\left[\frac{3-\nu}{8(1-\nu)}\rho^2 - \frac{1}{16}\rho^4\right]. \tag{15}$$

Then, after introducing eqns (14) and (15) into the right-hand side of eqn (11b) and integrating the equation, together with boundary condition (12a), for $\bar{w}$ again, the second approximation for the membrane deflection is found to be

$$\bar{w}^* = \frac{\beta}{4}(1-\rho^2) + \frac{\lambda\bar{w}_0}{16}(3-4\rho^2+\rho^4) + \varepsilon\bar{w}_0^3\left[\frac{3-\nu}{4(1-\nu)}(\rho^2-1) - \frac{1}{8}(\rho^4-1)\right] \tag{16}$$

or

$$\bar{w}^* = a_0 + a_2\rho^2 + a_4\rho^4 = -a_2(1-\rho^2) - a_4(1-\rho^4) \tag{17}$$

with

$$a_0 = \frac{\beta}{4} + \frac{3\lambda}{16}\bar{w}_0 - \frac{5-\nu}{8(1-\nu)}\varepsilon\bar{w}_0^3$$

$$a_2 = -\frac{\beta}{4} - \frac{\lambda}{4}\bar{w}_0 + \frac{3-\nu}{4(1-\nu)}\varepsilon\bar{w}_0^3$$

$$a_4 = \frac{\lambda}{16}\bar{w}_0 - \frac{1}{8}\varepsilon\bar{w}_0^3. \tag{18}$$

This expression comprises the contribution of all the non-linear terms in eqn (11b), and as a second approximation, we may expect it to give a fundamental description of the system behavior.

To finish the solution, parameter $\bar{w}_0$, which acts as an adjustable factor in the iteration, is decided by setting

$$\bar{w}^*(0) = \bar{w}_0 \tag{19}$$

which is an expedient consideration for consistency (or convergency). We note that the preceding setting (19) is only one of many ways of determining $\bar{w}_0$. From eqns (16) and (19), the governing equation for $\bar{w}_0$ is

$$\bar{w}_0 = \frac{\beta}{4} + \frac{3\lambda}{16}\bar{w}_0 - \varepsilon\left[\frac{5-\nu}{8(1-\nu)}\right]\bar{w}_0^3 \tag{20}$$

which also stands for the non-linear load–deflection relation being sought.

Continuing the process in a similar manner gives the second approximation for ψ, i.e.

$$\psi^* = \frac{1}{2}c_0\rho^2 - \frac{1}{16}a_2^2\rho^4 - \frac{1}{18}a_2a_4\rho^6 - \frac{1}{48}a_4^2\rho^8 \tag{21}$$

in which

$$c_0 = \frac{3-\nu}{4(1-\nu)}a_2^2 + \frac{5-\nu}{3(1-\nu)}a_2a_4 + \frac{7-\nu}{6(1-\nu)}a_4^2$$

and the third approximation for $\bar{w}$, i.e.

$$\bar{w}^{**} = b_0(1-\rho^2) + b_2(1-\rho^4) + b_4(1-\rho^6) + b_6(1-\rho^8) + b_8(1-\rho^{10}), \tag{22}$$

where

$$b_0 = \frac{1}{4}(\beta + a_0\lambda + 4a_2c_0\varepsilon),$$

$$b_2 = \frac{1}{16}(a_2\lambda - 2a_2^3\varepsilon + 16a_4c_0\varepsilon),$$

$$b_4 = \frac{1}{36}(a_4\lambda - 10a_2^2a_4\varepsilon),$$

$$b_6 = -\frac{1}{64}\left(\frac{40}{3}a_2a_4^2\varepsilon\right), \quad b_8 = -\frac{1}{100}\left(\frac{20}{3}a_4^3\varepsilon\right). \tag{23}$$

Seeing that expression (17) has already comprised the contribution of all the non-linear terms in eqns (11a, b), and that expression (22) is still an improvement on eqn (16), we may stop here and expect eqn (22) to give a fundamental description of the system behavior.

To finish the solution, parameter $\bar{w}_0$, which acts as an adjustable factor in the iteration, is decided by setting

$$\bar{w}^*(0) = \bar{w}^{**}(0), \tag{24}$$

which is an expedient consideration for consistency (or convergency). From eqns (16) and (22), the governing equation for $\bar{w}_0$ is then

$$\frac{\beta}{4}+\frac{3\lambda}{16}\bar{w}_0-\frac{5-\nu}{8(1-\nu)}\varepsilon\bar{w}_0^3 = b_0+b_2+b_4+b_6+b_8. \tag{25}$$

The central deflection of the membrane $\bar{w}^*$ (0) [or the same, $\bar{w}^{**}$ (0)] is of most interest. After having determined $\bar{w}_0$ from eqn (25), we finally have

$$\bar{w}^*(0) = \frac{\beta}{4}+\frac{3\lambda}{16}\bar{w}_0-\frac{5-\nu}{8(1-\nu)}\varepsilon\bar{w}_0^3. \tag{26}$$

We note that by following the process in the above manner, higher orders of approximation can be similarly obtained.

DISCUSSION AND CONCLUSION

Based on the foregoing results, four cases which are linked to the four different levels of problem handling can be examined successively.

(1) $\varepsilon = 0$, $\beta = 0$ (i.e. using linear theory and neglecting the membrane weight). By referring to eqns (18) and (23), it is seen that in this case

$$b_0 = \frac{3}{64}\lambda^2\bar{w}_0, \quad b_2 = -\frac{1}{64}\lambda^2\bar{w}_0$$
$$b_4 = \frac{1}{576}\lambda^2\bar{w}_0, \quad b_6 = b_8 = 0. \tag{27}$$

The third approximation (25) degenerates to

$$\frac{3}{16}\lambda\bar{w}_0 = \left(\frac{3}{64}-\frac{1}{64}+\frac{1}{576}\right)\lambda^2\bar{w}_0 \tag{28}$$

which yields

$$\bar{w}_0 = 0 \text{ [and therefore } \bar{w}^*(0) = 0] \tag{29}$$

or

$$\bar{w}_0 \text{ unconstrained [and } \bar{w}^*(0) \text{ unconstrained] for } \lambda = 5.68. \tag{30}$$

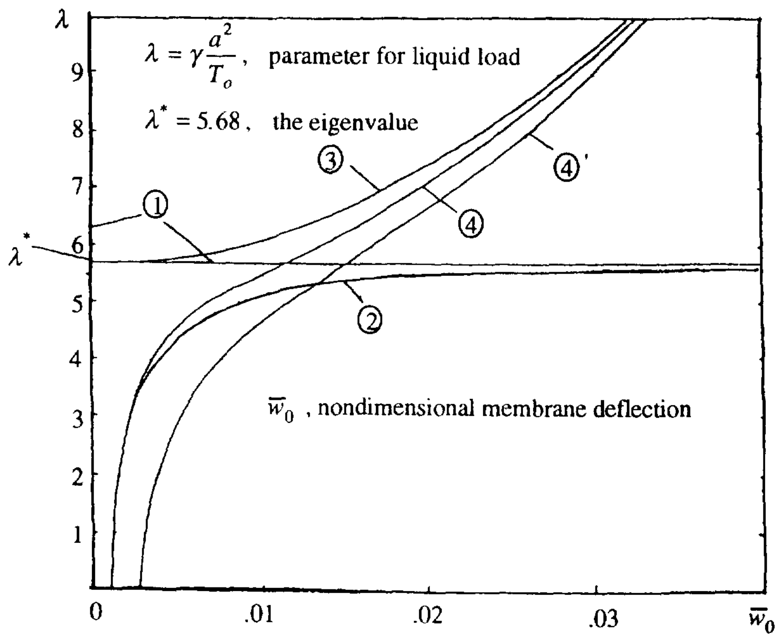

Fig. 2. Relations between membrane deflection and liquid load for four different cases.

This case gives unreasonable responses, with $\bar{w}(0)$ unconstrained [eqn (30)], $\lambda^* = 5.68$ being the eigenvalue and $\bar{w}_0 = 0$ [eqn (29)]. These responses are shown in Fig. 2 by lines 1.

We note that in the present case, the second approximation (20) for $\bar{w}_0$ yields $\bar{w}_0 = 0$ [hence $\bar{w}^*(0) = 0$], or $\bar{w}_0$ unconstrained [and hence $\bar{w}^*(0)$ unconstrained] for $\lambda^* = 5.33$.

(2) $\varepsilon = 0$, $\beta \neq 0$ (using linear theory and taking into account the membrane weight). Now, as $\beta \neq 0$, we have from eqns (18) and (23)

$$b_0 = \frac{\beta}{4} + \frac{1}{16}\lambda\beta + \frac{3}{64}\lambda^2\bar{w}_0$$

$$b_2 = -\frac{1}{64}\lambda\beta - \frac{1}{64}\lambda^2\bar{w}_0, \quad b_4 = \frac{1}{576}\lambda^2\bar{w}_0$$

$$b_6 = b_8 = 0. \tag{31}$$

The only solution for eqn (25) is

$$\bar{w}_0 = \frac{3\beta}{12 - \left(\frac{19}{9}\right)\lambda}. \tag{32}$$

Therefore eqn (26) yields

$$\bar{w}^*(0) = \frac{\beta}{4} + \frac{3\lambda}{16}\left(\frac{3\beta}{12 - \frac{19}{9}\lambda}\right). \tag{33}$$

This solution is plotted in Fig. 2 as line 2 (with $\beta = 0.004$). It is seen that non-zero deflection exists, which increases with the increasing of load parameter λ, and that very large deflection will happen when λ approaches the threshold value λ^* (5.68); but, as is known, linear theory is not applicable to analysis of large deflection, so line 2 can only have some meaning for λ being relatively small (i.e. for $\lambda \ll \lambda^*$).

In this case, the second approximation (20) yields

$$\bar{w}_0 = \frac{\beta}{4}\left(1-\frac{3\lambda}{16}\right) \tag{34}$$

and $\bar{w}^*(0) = \bar{w}_0$ [eqn (19)]. When this solution is plotted, it is very close to line 2 (Fig. 2) and hence the same conclusion can be made with $\lambda^* = 5.33$.

(3) $\varepsilon \neq 0$, $\beta = 0$ (non-linear theory and neglecting the membrane weight). In this case eqn (25) is too complicated for analytical treatment. From eqns (18) and (23), it is seen that this case has a trivial solution, $\bar{w}_0 = 0$. To search for any non-trivial solution, we put the third approximation (25) in the form

$$\bar{w}_0 = \frac{16}{3\lambda}\left[\frac{5-\nu}{8(1-\nu)}\varepsilon\bar{w}_0^3 + b_0 + b_2 + b_4 + b_6 + b_8\right] \tag{35}$$

then adopt a numerical iteration procedure. By doing so it is found that a non-trivial solution exists only for $\lambda > \lambda^*$ (5.68) as shown in Fig. 2 by line 3 (with $\varepsilon = 1000$, $\nu = 0.3$). This bifurcated solution (with bifurcation point λ^*) suggests that given $\lambda > \lambda^*$, even if the membrane weight is absent, a limited deflection may still occur. It is the same prediction as shown by Kerr and Coffin's (1990) non-linear analysis for a membrane strip. Thus, for problems of this type, a linear formulation is not suitable.

In this case the second approximation (20) yields

$$\bar{w}_0 = 0$$

or

$$\bar{w}_0 = \left\{\frac{\frac{3}{16}\lambda - 1}{(\varepsilon/8)[(5-\nu)/(1-\nu)]}\right\}^{1/2}, \quad \text{when} \quad \lambda > \lambda^* = 5.33. \tag{36}$$

Obviously, here the non-trivial $\bar{w}_0$ constitutes a bifurcated solution and the bifurcation point is exactly λ^* (5.33). When the λ–$\bar{w}_0$ curve based on eqn (36) is plotted, it coincides again as seen in the preceding case 2, with line 3, the third approximation, very well.

(4) $\varepsilon \neq 0$, $\beta \neq 0$ (non-linear theory and considering the effect of membrane weight). This is the case of most interest, the case which gives a more realistic picture. The same numerical procedure as in case 3 can be used for the solution of eqn (25). To observe the λ–$\bar{w}^*(0)$ relation so obtained, two typical curves, 4 ($\varepsilon = 1000$, $\nu = 0.3$, $\beta = 0.004$) and 4′ ($\varepsilon = 1000$, $\nu = 0.3$, $\beta = 0.01$), are drawn in Fig. 2. Inspecting curve 4, we see that by considering both the membrane weight and the non-linear character of the system, curve 4 bridges the gap between 2 and 3 as may be expected, and that the aforementioned eigenvalue (or bifurcation point) λ^* seems to be losing its specific meaning as there is no sudden change or "threshold" happening on the curve. It is also seen that the difference between 4 and 4′ caused by the two distinct but small values of β is significant only for the initial stage of the membrane deflection and that with an increase of load parameter λ, this difference becomes smaller and smaller.

In the present case, the second approximation (20) may be written as

$$\lambda = \left[\bar{w}_0 + \varepsilon\left(\frac{5-\nu}{8(1-\nu)}\right)\bar{w}_0^3 - \frac{\beta}{4}\right] \Big/ \frac{3\bar{w}_0}{16}. \tag{37}$$

When this λ–$\bar{w}_0$ relation is plotted, it is found that the difference between the curves based on eqn (25) and the relation (37) is negligibly small. Hence, only the two curves 4 and 4′ based on eqn (25) are plotted in Fig. 2.

With these results, we come to the following conclusions.

(1) For describing the exact behavior of a ponding system, non-linear (large deflection) analysis is necessary.

(2) Owing to the small weight of the membrane, ponding deflection will develop gradually with increasing load parameter. There is a stage with greatest change rate, but no sudden change happens.

(3) Such a small weight actually has no effect on the possible large deflection.

(4) The present iteration technique yields fast convergence and has the simplicity and effectiveness for solving non-linear membrane problems. It is well suited for practical usage.

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